How I Teach Word Problems

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Abstract

This article tells how the author teaches university freshmen to solve word problems and how it serves their intellectual development.

Key words: word problems, K-12, university, natural language.

As much as I remember, word problems always were present in the Russian mathematical education. Nobody questioned their importance and nobody considered them especially difficult. Already in elementary school children solve some simple word problems. As years pass, problems become more complex. In result, graduates of the better high schools have enough experience of solving word problems, so universities can go further. The same always seemed to be true of non-Russian parts of USSR and some other countries. It is only natural that when I grew up and started to teach, I used word problems a lot. Now, still more than ever, I believe that the ability to solve simple word problems practically coincides with the basic mathematical literacy. Besides simple, there are much more sophisticated word problems, so that you can contribute a lot to your students' proficiency by moving them from elementary to more advanced word problems, and hence - to professional mathematics. Simple word problems are still more useful for those who will never become professional mathematicians.

Since I came to America, I taught every year at a university level. Although I do not teach in K-12 now, I am well aware how my students are prepared. Many of my present students seem to have very little experience of solving word problems in high schools, so

I have to make this up. More than that, I found that even simple word problems are considered difficult here. For example, Mildred Johnson, who devoted a very useful book [1] to explanation in much detail how to solve the simplest word problems, writes in the preface: "There is no area in algebra which causes students as much difficulty as word problems." So I decided to describe how I use word problems when I teach college algebra at the University of the Incarnate Word. (College algebra is the first college-level course in mathematics in our university.)

I come to the classroom with a good supply of chalk and erasers. During class I invite four students to the blackboard at the same time, and dictate a problem to all the four at once. All four problems are the same except for one datum. For example, I tell them:

> **Problem 1**. Mary has a hundred coins in her piggy bank, some dimes, others quarters. Her total capital is...

While students write this, I realize that if all the coins were dimes, Mary would have ten dollars. If some dimes are replaces by quarters, it increases her capital by a multiple of fifteen cents. So I continue addressing each student in turn:

> ...thirteen, sixteen, nineteen, twenty-two dollars. How many dimes and how many quarters does Mary have?

Very soon students learn to understand what I mean and spend almost no time writing this down correctly.

I tell students that when they are at the board they are 'teachers', and must try to write clearly so that others can understand them. Whenever a student uses a variable, say X, I require her to write down what X means. For some it is a challenge and this challenge is very useful because it makes them think clearly. Sometimes I ask them to explain their solutions to the class in a loud voice, addressing their classmates. (Otherwise they tend to whisper and to address me or the blackboard.) I also tell them that only during tests and quizzes are they forbidden to communicate; at all other times they can and should help each other. For example, if a student at the board gets confused, her friend comes to help her, and this communication is a valuable experience for both of them. In the syllabus I wrote: "It is essential to understand that study is not a competition. Another student's success is not your failure and your classmate's failure is not your success."

Solutions usually take from five to ten minutes. All the still seated students are required to choose one of the four versions and work on it at the same time. They are willing to do this because they know I will allow them to use these notes during tests. I tell them: "If somebody at the blackboard makes a mistake, it is your mistake, because you should be checking each other. I have no time to check every calculation. Even if I see something wrong written on the board, I shall not tell you." (But actually I leave no mistake uncorrected.) From time to time groups of students form spontaneously in different parts of the room, discussing these problems. When all the four versions are solved, I ask if there are questions. I also make comments, explaining that one and the same problem can be solved in different ways, for example, using one or two variables or without algebra at all.

In this way I also correct many bad student habits. One is a careless and confused manner of writing. When adding two fractions or doing another arithmetical transformation, some students cover all the space with crossing lines and intermediate results. It becomes impossible to understand what is done, how it is done, whether it is correct and if not, where the mistake is. Another bad habit is 'immediate erasing': as soon as I tell a student that her solution as written on the blackboard is incorrect - sometimes even as soon as I say that I don't understand it, or just ask what it means - she immediately erases all of it, making further discussion impossible.

I remind my students that they must answer the questions which are asked, and we have to discuss what these questions mean. For example, many cannot figure out independently which quantity is meant when it is asked 'How far away ...' or 'When ...' or 'How long will it take ...' or 'How fast ...'. I have to teach my students that when making an equation they have to choose a specific unit for every quantity. For money it may be dollars or cents and, whatever they choose, they have to transform all money data into this unit. For time, it is usually hours or minutes, and all time data should be unified. I may also have to remind that there are 60 minutes in an hour, not 100. (Some students, when they need to transform 1/3 hour into minutes, grasp a calculator and come out with 33.3 minutes.)

I have to teach my students to organize data. An excellent technique is to place data into a chart. Let me show how we do it for the following problem.

> **Problem 2**. How much pure water should be added to 100 gram of 60% acid solution to make a 20% acid solution?

Most of my students cannot solve such a problem unless I give them a 'template' to organize the data. One way to do it is to place them in the following chart:

	given	added	total
mass in grams	100	X	$100 + X$
percentage of acid	$60\% = 0.6$	0%	$20\% = 0.2$
grams of acid	$0.6\ (100) = 60$	0	$0.2\ (100 + X)$

Since the quantity of acid does not change in the process, we may write an equation

$$60 + 0 = 0.2(100 + X),$$

solving which we obtain the answer: $X = 200$ grams. Let me list some of the mental operations which students need to perform in the course of this solution. (All of them are *nontrivial* and at the beginning of the course students make many mistakes.)

- To write appropriate and understandable names for rows and columns, such as 'given', 'added', 'total', 'volume' etc;

- To place data into appropriate boxes;

- To figure out that when two mixtures are poured together, the total mass equals the sum of masses of ingredients;

- To figure out that when two mixtures containing acid are poured together, the total amount of acid equals the sum of amounts of acid in the ingredients;

- To figure our that pure water contains 0 percent of acid;

- To notice that there are two expressions for the last box, which therefore are equal to each other, that an equation thus emerges, which can be solved to yeld the answer.

I also tell my students the following:

- Write carefully;

- Write every sign in a clear manner;

- Write every equation completely and clearly to make it easy to check;

- Make charts with care;

- Write the figures '0', '6' and '8' so they are distinguishable from each other;

- Write the figures '1' and '7' so they are distinguishable from each other;

- Don't write the letter 'l' just as a vertical bar, it must be distinguishable from the figure '1',

and many other things which are taken for granted by those who had good teachers in childhood. Is all this mathematics? The answer, of course, depends on how we define mathematics. But in any case all this *needs* to be taught, otherwise there will be no mathematics.

Of course, most students cannot work all this out independently. I have to tell them, and there is nothing reprehensible about it. Even this course is too difficult for many students. Some do not take college algebra at all. Some of those who take college algebra try to reduce solving problems to still more simple rules. And there is nothing mysterious about these difficulties. Remember that algebra is not in our genes; it is in our culture. Transmission of culture needs explanations. If explanations have never been given, a person gets lost: she puts data into wrong boxes, confuses the relations between distance, time and speed etc. This is not stupidity or inferiority; this is lack of schooling.

One challenge to which I subject the strongest of my students are 'impossible' problems. Suppose I have four students at the blackboard and dictate:

> **Problem 3**. At noon Bob went out jogging at 5 mph. An hour later Ana went on the same route on her bicycle and caught him up (*addressing each student in turn*) 4, 6, 8, 10 miles away. What was Ana'a speed?

All four students go about solving the problem in a similar way. After trials and tribulations, three come up with an acceptable answer; but the first one comes up with a negative answer. I ask all the class to help her. We check the calculations and see that there is no mistake. Sometimes one of the students comes up with the right explanation, sometimes I have to observe that when Ana starts, Bob is already five miles away. Anyway, I make my students aware that they must be able to check the results of their calculations against common sense and to conclude “no answer” when necessary.

Some of my top priorities as a teacher of mathematics are:

- To make students understand and use their mother tongue better to convey exact information;

- To develop students’ ability to represent information in ways useful for stating and solving problems;

- To teach students to translate between different modes of representation: English, algebra, charts, graphs;

- To improve their manners (understandable writing, fruitful communication, including ability to explain and to understand an explanation).

To achieve this, students need to be given definite and exact ‘rules of the game’. What is given, what is asked and how to tell a right answer from a wrong one must be clear. Word problems like those described above are very suitable for this.

The following example illustrates this. Once a student asked my help in dealing with a problem similar to Problem 2 . I answered: ‘Make a chart’. ‘Is it your requirement?’ - asked the student impatiently. I said - ‘No, but since you say that you are lost, make a chart.’ The student endeavored to make a chart as if she were doing a favor to an old pedant and solved the problem. I said: ‘Now let me tell you something about teaching.

You expected me to help you. Did I?' She said: 'Yes, you did.' 'But I told you nothing.' 'You told me to make a chart.' Solving word problems helps students to organize their ideas.

In the course of this kind of teaching I came to the conclusion that **teaching to understand and intelligently use the natural language**, in the present case English, is one of the most urgent functions of mathematical education. In this sense public mathematical education is at once much less and much more than teaching mathematics. Less, because most students will never reach the level of professional mathematicians. More, because mathematics is an essential part of modern civilization, which is not in our genes and which takes schooling to transmit to the next generation.

You may ask: "Why do we need to teach students their mother tongue if all of them already know it?" But there are different ways and levels of knowing one's native language. It takes only superficial knowledge to exchange casual greetings: "Hi! - Hi! - How are you? - Just fine. - Take care." It takes much more to comprehend a text describing some system of formal relations. Sister Teresa Grabber, who teaches remedial algebra, observed in a discussion: 'When my students cannot solve a word problem, I discuss with them why they cannot, and we conclude that they cannot read.' I replied: 'Well, you don't mean this literally.' She agreed: 'No, I don't. I mean lack of comprehension.'

I often ask students to explain solutions to each other and I think that this is a very valuable experience for them. When students make gestures imitating such 'realities' as moving cars or current in a river, they make abstractions almost visible and touchable. I say 'abstractions' because these cars and current are not real and this is their great advantage. Since riders, pumps and other 'realities' mentioned in word problems are cleaned of irrelevant details, they serve as semi-abstractions, still understandable for novices. This makes word problems an excellent breeding-ground for initial study of mathematics and science. After discussions, my students write equations, in which every sign is rooted in their visual and motor experiences. The joy of understanding which they feel is the most appropriate reward for doing mathematics. This reward actually coincides with the purpose and results of teaching.

Conclusion. Simple traditional word problems are indispensable for the public mathematical education. Their main function is to serve the initial development of abstract thinking, not to be applied to practice in the literal sense. Many high school graduates cannot solve even simple word problems and universities have to make this up. It is possible and desirable to teach to solve word problems much earlier, certainly not later than in high school.

All responsibility for this paper is mine, however I thank all of those who kindly edited its early version, especially Madge Goldman and Ralph A. Raimi.

References

[1] Johnson, Mildred. 1992. *How to solve word problems in algebra. A solved problem approach.* Updated First Edition. McGraw-Hill.