THE MATHEMATICS OF PHYSICAL QUANTITIES

PART I : MATHEMATICAL MODELS FOR MEASUREMENT

Whitney Hassler, *Institute for Advanced Study*

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## INTRODUCTION

1. **Purpose of this paper**. To set up a physical theory, one constructs a mathematical model, and considers its relation to certain aspects of the physical world. There is a variety of models associated with the concept of measurement. Certain systems of numbers are important; for instance:

***N***, the natural numbers; ***R+***, the positive reals;
***Q+***, the positive rationals; ***R***, the reals.

Commonly one takes ***R+*** or ***R*** as a model for measurement. There are disadvantages in this, however. These models contain a specific number 1, and there is no natural way of putting this number in correspondence with a particular measurement; moreover, the models contain an operation of multiplication, with no natural physical counterpart.

Let us consider the problem of choosing a model ***M*** for masses. An object ***A*** has a certain property which we call its "mass" ; why not let this property itself be an element of the model? As far as the structure of the model is concerned, we need not theorize on what "mass" really is; we need merely give it certain properties in the model. For instance, if we have distinct objects ***A*** and ***B***, with masses $m_A$ and $m_B$, we may think of the objects as forming a single object C; its mass $m_C$ should then be $m_A+m_B$. Therefore ***M*** should contain an operation of addition, and any further properties we choose.

The two types of models that best fit in most situations we shall call "rays" and "birays." A ray (like a half line) is used for positive measurements, and a biray ( like an oriented line with starting point ), for measurements of quantities both positive and negative. It turns out that numbers appear in a natural way as operators on the model (see the next section).

In this paper we set up the theory of rays and birays; the real number system is constructed along with the models in a natural manner. In fact, this approach gives a simple and elegant way of introducing the reals and finding their basic properties.

We shall not give applications of the models in the body of the paper; some remarks on the subject will be made in this introduction. In this connection, see Part II, which will appear in the next issue of this MONTHLY.

2. **Numbers as operators on the models**. If we choose a stick of length *l*, and wish to use it to measure another stick, we lay out the first stick along the second several times; we thus form $l+l$, $l+l+l$, and so on. We also call these lengths $2l$, $3l$, - - - . Thus ***N*** appears as a natural set of operators on our model. For masses, there is a different physical process of addition; but again we may use $2m = m + m$ and so on, with the same set ***N*** of numbers.

If we have a stick, whose length we call *1*, and we find a shorter stick, of length $l'$, such that $3\,1' = 1$, then we wish to give an expression of $l'$ in terms of *1*. It is natural to set $1' = (1/3)l$. We may now set $2l' = (2 / 3)1$, and thus introduce ***Q+*** as operators. Finally, if our model has a certain completeness property, we may enlarge ***Q+*** to ***R+*** as operator system, and if we have negative quantities, we may enlarge ***R+*** to ***R***.

3. **Working in the model**. Various properties of a model and its operations have obvious meaning in the applications. For instance we have distributive and associative laws:

5 cakes + 2 cakes = (5 + 2)cakes = 7 cakes
2 yd = 2(3ft) = (2 x3)ft = 6ft.

The fact that "2 yd" and "6 ft" name the same element of the model enables us to say they are equal; there is no need for such mysterious phrases as "2 yd measures the same as 6 ft."

4. **The use of units**. If we wish to use ***R+*** as a model for measuring (positive) lengths, we must decide which length *1* corresponds to 1; this length will then be called our "unit length." The more natural model is a ray ***L*** (on which ***R+*** operates) ; since the elements of ***L*** themselves are "lengths," the above question does not arise.

If we choose a length $l_0 \in$ ***L***, and compare other lengths with it, we may call, $l_0$ our "unit"; this serves merely to remind us that $l_0$ is being kept fixed for a period. Suppose we now find certain other lengths, for instance 5 $l_0$, 2 $l_0$, 7 $l_0$. If "-e wish to shorten our notations, and call these lengths 5, 2, 7, we are then replacing ***L*** by ***R+***. We can then say "the length 5 really means the length 5 $l_0$." More awkwardly, one could say "the length is 5 when measured in terms of $lo$."

Suppose we wish to "change units," say from ft to in. Then since, for any $a \in$ ***R+***, $a$ ft = $a$(12 in) =12 $a$ in, we would replace "the length a" by "the length 12a." If any problems about units arise, they are at once resolved by going back to the explicit phrase "$a$ ft."

5. **The postulational treatment.** Though all rays (and all birays) have the same structure, one may wish to use several rays in a single investigation. For instance, in mechanics, one uses separate rays ***M***, ***L***, ***T*** for measurement of mass, length and time. (We study structures containing several rays in Part II.) Hence we introduce our models postulationally; the definitions show whether or not a given structure is a ray or a biray. However, only a single structure ***R*** (or one of its subsets) is needed for operators; hence we introduce ***R*** constructively. (We give the characterization of ***R*** as a complete ordered field at the end.)

A basic theorem in the subject is an *isomorphism theorem*; a homomorphism of one ray into another is necessarily an isomorphism onto, and has certain additional properties (and similarly for birays). This theorem is a great aid in setting up the theory; in particular, with its use, multiplication in ***R+*** and in ***R*** is introduced and its properties derived with a minimal effort.

The postulates used for rays and birays are few in number and simple in character, and correspond to simple experimental phenomena.

6. **Other models**. We introduce only the most important models. With the real numbers at our disposal, and the facts about rays and birays, other similar models are easily studied. For instance, in measuring masses, one wishes to allow the mass zero (not present in a ray). This extra element may be introduced and related to the remaining elements in the obvious manner.

A model of a somewhat different nature is an oriented affine one-dimensional space ***T****; this is the natural model for instance for moments in time (or positions on a line). There is a corresponding biray ***T*** of translations of ***T****; this is the natural model for intervals of time (or directed lengths). We do not consider models including for instance 3-dimensional space; the term "measurement" is not the best term here.

If the measures of some type of quantity form a progression, as in counting, it is natural to use ***N*** for a model. However, if several such types of quantities are considered together, it is better to use several isomorphic models. For a plebeian illustration, suppose there will be six children at a party. We wish each to have two balloons and three cookies. What is the total supply needed? The answer is:

$$6(2 \text{ bl} + 3 \text{ ck}) = 6(2 \text{ bl}) + 6(3 \text{ ck}) = 12 \text{ bl} + 18 \text{ ck}.$$

7. **A finite model**. We have no infinite sets available in our environment. What happens if we have a set G with a large number $n$ of elements, called $l', 2' \ldots, n'$, and wish it to approximate to ***N***? We could define $a' + b'$ to be $(a+b)'$, or $n'$ if $n < a+b$. Note that we may set $a'b' = ab'$ ; now ***G*** operates on itself, thus defining multiplication in ***G***. We find that these operations are commutative and associative, and the distributive laws hold. However, the cancellation laws fail.

We give an instance from everyday life. Helen is setting the table for lunch for four; she places two spoons at each place. Mother answers the doorbell; it is Mr. and Mrs. Jones. Perhaps they will stay; Helen needs 4 more spoons. There are only two left in the drawer, so Helen puts them out. (She thus makes 8s+4s = 10s.) Hearing the visitors say goodbye, Helen thinks, take away four spoons. She then realizes that, actually, she must take away only two. In her model, 8s + 4s = 8s + 2s.