# Geometry and Imagination

David Hilbert and S. Cohn-Vossen[1]

## Preface by David Hilbert

In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.

As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that *intuitive* understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry.

In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual, intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof, without necessarily entering into the details connected with the strict definitions of concepts and with the actual calculations. For example, the proof of the fact that a sphere with a hole can always be bent - no matter how small the hole - or of the fact that two different toroidal surfaces can not in general be wrapped onto each other conformally, can be treated in such a fashion that even one who does not wish to follow the details of the analytical arguments, may still gain an insight into how and why the proof works.

In this manner, geometry being as many-faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains. Thus a presentation of geometry in large brushstrokes, so to speak, and based on the approach through visual intuition, should contribute to a more just appreciation of mathematics by a wider range of people than just the specialists. For it is true, generally speaking, that mathematics is not a popular subject, even though its importance may be generally conceded. The reason for this is to be found in the common superstition that mathematics is but a continuation, a further development, of the fine art of arithmetic, of juggling with numbers. Our book aims to combat that superstition, by offering, instead of formulas, figures that may be looked at and that may easily be supplemented by models which the reader can construct. This book was written to bring about a greater enjoyment of mathematics, by making it easier for the reader to penetrate to the

[1] David Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Julius Springer, Berlin, 1932.
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essence of mathematics without having to weight himself down under a laborious course of studies.

With aims like these to strive after, there could be no question of strict systematic arrangement or of completeness, nor was it possible to treat individual topics exhaustively. Also, it was impossible to assume the same amount of mathematical training on the reader's part as a prerequisite for all sections of the book; while the presentation is for the most part quite elementary, there are nevertheless some beautiful geometric investigations which can be fully explained only to those with a certain amount of training if tiresome length of presentation is to be avoided.

The appendices to the various chapters all assume a certain amount of knowledge for their understanding ; they are throughout supplements to, and not explanations of, the main text.
The various branches of geometry are all interrelated closely and quite often unexpectedly. This shows up in many places in this book. Even so, because of the great diversity of the material treated, it was necessary to make each chapter more or less self-contained, and to avoid making the later chapters dependent for their understanding on a complete acquaintance with the earlier ones. We hope that, by making a few minor repetitions, we have rendered each chapter taken by itself - occasionally even an individual section taken by itself - understandable and interesting. We want to take the reader on a leisurely walk, as it were, in the big garden that is geometry, so that each may pick for himself a bouquet to his liking.

The basis for this book was a course of lectures, given four times weekly, called *Anschauliche Geometrie,* which I gave at Göttingen in the winter of 1920-21 and for which W. Rosemann worked out notes. In essence, the outline and contents of that course have been retained for this book, but S. Cohn-Vossen has re-worked many details, and has supplemented the material in quite a few places.

The line diagrams have all been drawn by K. H. Naumann and H. Bödeker (Göttingen). The photographic pictures were taken by W. Jentzsch (Göttingen), and the models he photographed belong to the collection of the Göttingen Mathematical Institute. The following have read the manuscript and proofs and made many valuable suggestions: W. Fenchel, H. Lewy, H. Schwerdtfeger, H. Heesch, and especially A. Schmidt. The final arrangement of the book has been S. Cohn-Vossen's responsibility.

**David Hilbert**

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